

ON THE STABILITY OF AN ADDITIVE-QUARTIC SET-VALUED FUNCTIONAL EQUATION

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ABSTRACT. In this paper, I investigate a stability of the following set-valued functional equation

$$\begin{aligned} f(x+3y) \oplus 10f(x+y) \oplus 7f(-x) \oplus 5f(x-y) \\ = 5f(x+2y) \oplus f(x) \oplus f(2x) \oplus f(x-2y) \end{aligned}$$

in the sense of P. Găvruta.

1. Introduction

In 1940, Ulam [14] first posed a question about the stability of group homomorphisms. In 1941, Hyers [4] provided a positive answer to this question about additive mappings between Banach spaces. Since then many mathematicians have dealt with this problem [3, 8, 9, 13].

A solution of the functional equation

$$f(x+y) - f(x) - f(y) = 0$$

is called an additive mapping and a solution of the functional equation

$$f(x+2y) - 4f(x+y) + 6f(x) - 4f(x-y) + f(x-2y) - 24f(y) = 0$$

is called a quartic mapping. Now I consider the following functional equation

$$(1.1) \quad \begin{aligned} f(x+3y) - 5f(x+2y) + 10f(x+y) - f(x) + 7f(-x) - f(2x) \\ + 5f(x-y) - f(x-2y) = 0. \end{aligned}$$

The mapping $f(x) = ax^4 + bx$ is a solution of this functional equation, where a, b are real constants. A mapping f is called an additive-quartic mapping if f is represented by the sum of an additive mapping and a quartic mapping.

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Lu and Park [7] initiated the study of the Ulam stability of set-valued functional equations as a generalization of the stability of single-valued functional equations. Along the way, many authors have studied the stability problem of set-value functional equations of various types [2, 5, 6, 10, 11].

The following terminologies used in this paper will be adopted from the article of Kenary et al. [6].

Throughout this paper, unless otherwise specified, let X be a real vector space and Y be a Banach space with the norm $\|\cdot\|_Y$. We denote by $C_b(Y)$, $C_c(Y)$ and $C_{cb}(Y)$ the set of all closed bounded subsets of Y , the set of all closed convex subsets of Y and the set of all closed convex bounded subsets of Y , respectively. For the nonempty sets $A, B \subset Y$ and $\lambda \in \mathbb{R}$, let $A + B, \lambda A$ be the sets defined as follows

$$A + B = \{a + b | a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a | a \in A\}.$$

Furthermore, for the subsets $A, B \in C_c(Y)$, we write $A \oplus B = \overline{A + B}$, where $\overline{A + B}$ denotes the closure of $A + B$. Generally, we can obtain that

$$\lambda A + \lambda B = \lambda(A + B) \text{ and } (\lambda + \mu)A \subset \lambda A + \mu A$$

for arbitrary $\lambda, \mu \in \mathbb{R}^+$. In particular, if A is convex, then we have $(\lambda + \mu)A = \lambda A + \mu A$. For $A, B \in C_b(Y)$, the Hausdorff distance between A and B is defined by

$$h(A, B) := \inf\{\varepsilon > 0 | A \subset B + \varepsilon \overline{S_1}, B \subset A + \varepsilon \overline{S_1}\},$$

where S_1 denotes the closed unit ball in Y , i.e. $S_1 = \{y \in Y | \|y\|_Y \leq 1\}$. I easily know that

$$h(A, B) = h(-A, -B)$$

for all $A, B \in C_c(Y)$ from the definition of h . Since Y is a Banach space, it is proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup [1]. Rådström [12] proved that $(C_{cb}(Y), \oplus, h)$ can be isometrically embedded in a Banach space. The following are some properties of the Hausdorff distance.

LEMMA 1.1. (Castaing and Valadier [1]). *For any $A_1, A_2, B_1, B_2, C \in C_{cb}(Y)$ and $\lambda \in \mathbb{R}^+$, the following expressions hold*

- (i) $h(A_1 \oplus A_2, B_1 \oplus B_2) \leq h(A_1, B_1) + h(A_2, B_2)$;
- (ii) $h(\lambda A_1, \lambda B_1) = \lambda h(A_1, B_1)$;
- (iii) $h(A_1 \oplus C, B_1 \oplus C) = h(A_1, B_1)$.

In particular, from (i) and (iii) in Lemma 1.1, I have

$$h(A, C) = h(A \oplus B, B \oplus C) \leq h(A, B) + h(B, C)$$

for any $A, B, C \in C_{cb}(Y)$.

The main purpose of this paper is to establish the stability of the following additive-quartic set-valued functional equation

$$(1.2) \quad \begin{aligned} f(x + 3y) \oplus 10f(x + y) \oplus 7f(-x) \oplus 5f(x - y) \\ = 5f(x + 2y) \oplus f(x) \oplus f(2x) \oplus f(x - 2y) \end{aligned}$$

in the sense of P. Găvruta [3]. Namely, starting from the given mapping f that approximately satisfies the functional equation (1.2), a solution F of the functional equation (1.2) is explicitly constructed by using the formula either

$$F(x) = \lim_{n \rightarrow \infty} \left(\frac{1 + 8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1 - 8^n}{2 \cdot 16^n} f(-2^n x) \right)$$

or

$$F(x) = \lim_{n \rightarrow \infty} \left(\frac{16^n + 2^n}{2} f(2^{-n} x) \oplus \frac{16^n - 2^n}{2} f(-2^{-n} x) \right)$$

which approximates the mapping f .

2. Stability of the additive-quartic set-valued functional equation (1.2)

Now I will establish the stability of the set-valued functional equation (1.2) in the sense of P. Găvruta by employing the direct method.

THEOREM 2.1. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that*

$$(2.1) \quad \Phi(x, y) = \sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} < \infty$$

for all $x, y \in X$. Suppose that $f : X \rightarrow C_{cb}(Y)$ is a mapping satisfying

$$(2.2) \quad \begin{aligned} h(f(x + 3y) \oplus 10f(x + y) \oplus 7f(-x) \oplus 5f(x - y), \\ 5f(x + 2y) \oplus f(x) \oplus f(2x) \oplus f(x - 2y)) \leq \varphi(x, y) \end{aligned}$$

for all $x, y \in X$. Then there exists a unique mapping $F : X \rightarrow C_{cb}(Y)$ such that F is a solution of the functional equation (1.2) and

$$(2.3) \quad h(f(x), F(x)) \leq \sum_{n=0}^{\infty} \left(\frac{8^{n+1} + 1}{2 \cdot 16^{n+1}} \varphi(2^n x, 0) + \frac{8^{n+1} - 1}{2 \cdot 16^{n+1}} \varphi(-2^n x, 0) \right)$$

for all $x \in X$. In particular, F is represented by

$$(2.4) \quad F(x) = \lim_{n \rightarrow \infty} \left(\frac{1 + 8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1 - 8^n}{2 \cdot 16^n} f(-2^n x) \right)$$

for all $x \in X$.

Proof. Setting $y = 0$ in (2.2), I have

$$(2.5) \quad \begin{aligned} h(9f(x) \oplus 7f(-x), f(2x)) &= h(16f(x) \oplus 7f(-x), f(2x) \oplus 7f(x)) \\ &\leq \varphi(x, 0) \end{aligned}$$

for all $x \in X$. Replacing x by $2^n x$ in (2.5) and dividing both sides by $2 \cdot 16^{n+1}$, from the equality (ii) in Lemma 1.1, we get

$$h\left(\frac{9f(2^n x)}{2 \cdot 16^{n+1}} \oplus \frac{7f(-2^n x)}{2 \cdot 16^{n+1}}, \frac{f(2^{n+1}x)}{2 \cdot 16^{n+1}}\right) \leq \frac{\varphi(2^n x, 0)}{2 \cdot 16^{n+1}}$$

for all $x \in X$ and $n \in \mathbb{N}$. By the above inequality and the inequality (i) in Lemma 1.1, I have

$$\begin{aligned} &h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x), \right. \\ &\quad \left. \frac{1+8^{n+1}}{2 \cdot 16^{n+1}} f(2^{n+1}x) \oplus \frac{1-8^{n+1}}{2 \cdot 16^{n+1}} f(-2^{n+1}x)\right) \\ &\leq h\left(\frac{9(1+8^{n+1})f(2^n x)}{2 \cdot 16^{n+1}} \oplus \frac{7(1+8^{n+1})f(-2^n x)}{2 \cdot 16^{n+1}}, \frac{1+8^{n+1}}{2 \cdot 16^{n+1}} f(2^{n+1}x)\right) \\ &\quad + h\left(\frac{9(1-8^{n+1})f(-2^n x)}{2 \cdot 16^{n+1}} \oplus \frac{7(1-8^{n+1})f(2^n x)}{2 \cdot 16^{n+1}}, \frac{1-8^{n+1}}{2 \cdot 16^{n+1}} f(-2^{n+1}x)\right) \\ &\leq \frac{8^{n+1}+1}{2 \cdot 16^{n+1}} \varphi(2^n x, 0) + \frac{8^{n+1}-1}{2 \cdot 16^{n+1}} \varphi(-2^n x, 0) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. From the above inequality and the property $h(A, C) \leq h(A, B) + h(B, C)$, we obtain

$$(2.6) \quad \begin{aligned} &h\left(f(x), \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)\right) \\ &\leq \sum_{k=0}^{n-1} h\left(\frac{1+8^k}{2 \cdot 16^k} f(2^k x) \oplus \frac{1-8^k}{2 \cdot 16^k} f(-2^k x), \right. \\ &\quad \left. \frac{1+8^{k+1}}{2 \cdot 16^{k+1}} f(2^{k+1}x) \oplus \frac{1-8^{k+1}}{2 \cdot 16^{k+1}} f(-2^{k+1}x)\right) \\ &\leq \sum_{k=0}^{n-1} \left(\frac{8^{k+1}+1}{2 \cdot 16^{k+1}} \varphi(2^k x, 0) + \frac{8^{k+1}-1}{2 \cdot 16^{k+1}} \varphi(-2^k x, 0) \right) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. In particular, if $\varphi(x, y) := 0$ i.e., f is a solution of the functional equation od (1.2), then I know that

$$(2.7) \quad f(x) = \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)$$

for all $x \in X$ and $n \in \mathbb{N}$. Now I claim that the sequence $\{\frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$. Indeed, for all $m, n \in \mathbb{N}$, by (2.6), one can shows that

$$\begin{aligned} & h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x), \frac{1+8^{m+n}}{2 \cdot 16^{m+n}} f(2^n x) \oplus \frac{1-8^{m+n}}{2 \cdot 16^{m+n}} f(-2^n x)\right) \\ & \leq \sum_{k=n}^{m+n-1} \left(\frac{8^{k+1}+1}{2 \cdot 16^{k+1}} \varphi(2^k x, 0) + \frac{8^{k+1}-1}{2 \cdot 16^{k+1}} \varphi(-2^k x, 0) \right) \\ & \leq \sum_{k=n}^{m+n-1} \frac{\varphi(2^k x, 0) + \varphi(-2^k x, 0)}{2^k} \end{aligned}$$

for all $x \in X$. From the condition (2.1), it follows that the right side of the above inequality tends to zero as $n \rightarrow \infty$. So the sequence $\{\frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)\}$ is a Cauchy sequence. Therefore, from the completeness of $C_{cb}(Y)$, I can define a set-valued mapping $F : X \rightarrow (C_{cb}(Y))$ represented by (2.4) for all $x \in X$.

Next, I will show that F satisfies the set-valued functional equation (1.1). From the properties $h(A, D) \leq h(A, B) + h(B, C) + h(C, D)$ and $h(A_1 \oplus A_2 \oplus \dots \oplus A_n, B_1 \oplus B_2 \oplus \dots \oplus B_n) \leq h(A_1, B_1) + h(A_2, B_2) + \dots + h(A_n, B_n)$, I have

$$\begin{aligned} & h(F(x+3y) \oplus 10F(x+y) \oplus 7F(-x) \oplus 5F(x-y), \\ & \quad 5F(x+2y) \oplus F(x) \oplus F(2x) \oplus F(x-2y)) \\ & \leq h\left(\frac{5(1+8^n)}{2 \cdot 16^n} f(2^n(x+2y)) \oplus \frac{5(1-8^n)}{2 \cdot 16^n} f(-2^n(x+2y)) \oplus \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \right. \\ & \quad \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x) \oplus \frac{1+8^n}{2 \cdot 16^n} f(2^{n+1} x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^{n+1} x) \\ & \quad \oplus \frac{1+8^n}{2 \cdot 16^n} f(2^n(x-2y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x-2y)), \\ & \quad \left. 5F(x+2y) \oplus F(x) \oplus F(2x) \oplus F(x-2y)\right) \end{aligned}$$

$$\begin{aligned}
& +h\left(F(x+3y) \oplus 10F(x+y) \oplus 7F(-x) \oplus 5F(x-y), \right. \\
& \quad \frac{1+8^n}{2 \cdot 16^n} f(2^n(x+3y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x+3y)) \oplus \frac{10(1+8^n)}{2 \cdot 16^n} f(2^n(x+y)) \\
& \quad \oplus \frac{10(1-8^n)}{2 \cdot 16^n} f(-2^n(x+y)) \oplus \frac{7(1+8^n)}{2 \cdot 16^n} f(-2^n x) \oplus \frac{7(1-8^n)}{2 \cdot 16^n} f(2^n x) \\
& \quad \left. \oplus \frac{5(1+8^n)}{2 \cdot 16^n} f(2^n(x-y)) \oplus \frac{5(1-8^n)}{2 \cdot 16^n} f(-2^n(x-y))\right) \\
& +h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^n(x+3y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x+3y)) \oplus \frac{10(1+8^n)}{2 \cdot 16^n} f(2^n(x+y)) \right. \\
& \quad \oplus \frac{10(1-8^n)}{2 \cdot 16^n} f(-2^n(x+y)) \oplus \frac{7(1+8^n)}{2 \cdot 16^n} f(-2^n x) \oplus \frac{7(1-8^n)}{2 \cdot 16^n} f(2^n x) \\
& \quad \oplus \frac{5(1+8^n)}{2 \cdot 16^n} f(2^n(x-y)) \oplus \frac{5(1-8^n)}{2 \cdot 16^n} f(-2^n(x-y)), \\
& \quad \frac{5(1+8^n)}{2 \cdot 16^n} f(2^n(x+2y)) \oplus \frac{5(1-8^n)}{2 \cdot 16^n} f(-2^n(x+2y)) \\
& \quad \oplus \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x) \oplus \frac{1+8^n}{2 \cdot 16^n} f(2^{n+1}x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^{n+1}x) \\
& \quad \left. \oplus \frac{1+8^n}{2 \cdot 16^n} f(2^n(x-2y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x-2y))\right) \\
& \leq 5h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^n(x+2y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x+2y)), F(x+2y)\right) \\
& \quad + h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x), F(x)\right) \\
& \quad + h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^{n+1}x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^{n+1}x), F(2x)\right) \\
& \quad + h\left(\frac{1+8^n}{2 \cdot 16^n} f(2^n(x-2y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x-2y)), F(x-2y)\right) \\
& \quad + h\left(F(x+3y), \frac{1+8^n}{2 \cdot 16^n} f(2^n(x+3y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x+3y))\right) \\
& \quad + 10h\left(F(x+y), \frac{1+8^n}{2 \cdot 16^n} f(2^n(x+y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x+y))\right) \\
& \quad + 7h\left(F(-x), \frac{1+8^n}{2 \cdot 16^n} f(-2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(2^n x)\right) \\
& \quad + 5h\left(F(x-y), \frac{1+8^n}{2 \cdot 16^n} f(2^n(x-y)) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n(x-y))\right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2^n} h(f(2^n(x+3y)) \oplus 10f(2^n(x+y)) \oplus 7f(-2^n x) \oplus 5f(2^n(x-y)), \\
& \quad 5f(2^n(x+2y)) \oplus f(2^n x) \oplus f(2^{n+1}x) \oplus f(2^n(x-2y))) \\
& + \frac{1}{2^n} h(f(-2^n(x+3y)) \oplus 10f(-2^n(x+y)) \oplus 7f(2^n x) \oplus 5f(-2^n(x-y)), \\
(2.8) \quad & \quad 5f(-2^n(x+2y)) \oplus f(-2^n x) \oplus f(-2^{n+1}x) \oplus f(-2^n(x-2y)))
\end{aligned}$$

for all $x, y \in X$. Since $\lim_{n \rightarrow \infty} h(F(x), \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)) = 0$ and

$$\begin{aligned}
& \frac{1}{2^n} h(f(2^n(x+3y)) \oplus 10f(2^n(x+y)) \oplus 7f(-2^n x) \oplus 5f(2^n(x-y)), \\
& \quad 5f(2^n(x+2y)) \oplus f(2^n x) \oplus f(2^{n+1}x) \oplus f(2^n(x-2y))) \\
& + \frac{1}{2^n} h(f(-2^n(x+3y)) \oplus 10f(-2^n(x+y)) \oplus 7f(2^n x) \oplus 5f(-2^n(x-y)), \\
& \quad 5f(-2^n(x+2y)) \oplus f(-2^n x) \oplus f(-2^{n+1}x) \oplus f(-2^n(x-2y))) \\
& \leq \frac{1}{2^n} \varphi(2^n x, 2^n y) + \frac{1}{2^n} \varphi(-2^n x, -2^n y) \\
& \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

all terms on the right hand side of the inequality (2.8) tend to zero as $n \rightarrow \infty$. This implies that F is an additive-quartic set-valued mapping. Moreover, letting $n \rightarrow \infty$ in (2.6), I get the desired inequality (2.3). To prove the uniqueness of F , assume that a set-valued mapping F' is another solution of the functional equation of (1.2) satisfying the inequality (2.3). Then the equality $F'(x) = \frac{1+8^n}{2 \cdot 16^n} F'(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} F'(-2^n x)$ follows from (2.7). Thus I have

$$\begin{aligned}
& h\left(F'(x), \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)\right) \\
& \leq h\left(\frac{1+8^n}{2 \cdot 16^n} F'(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} F'(-2^n x), \right. \\
& \quad \left. \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x)\right) \\
& \leq \frac{1+8^n}{2 \cdot 16^n} h(F'(2^n x), f(2^n x)) + \frac{8^n-1}{2 \cdot 16^n} h(F'(-2^n x), f(-2^n x))
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1+8^n}{2 \cdot 16^n} \sum_{k=0}^{\infty} \left(\frac{8^{k+1}+1}{16^{k+1}} \varphi(2^{k+n}x, 0) + \frac{8^{k+1}-1}{16^{k+1}} \varphi(-2^{k+n}x, 0) \right) \\
&\leq \frac{1}{2^n} \sum_{k=0}^{\infty} \left(\frac{1}{2^k} \varphi(2^{k+n}x, 0) + \frac{1}{2^k} \varphi(-2^{k+n}x, 0) \right) \\
&\leq \sum_{k=n}^{\infty} \left(\frac{1}{2^k} \varphi(2^kx, 0) + \frac{1}{2^k} \varphi(-2^kx, 0) \right).
\end{aligned}$$

for all $x \in X$. It is easy to see from the condition (2.1) that all terms on the right hand side of the above inequality tend to zero as $n \rightarrow \infty$, i.e. $F'(x) = \lim_{n \rightarrow \infty} \frac{1+8^n}{2 \cdot 16^n} f(2^n x) \oplus \frac{1-8^n}{2 \cdot 16^n} f(-2^n x) = F(x)$ for all $x \in X$. This completes the proof of this theorem. \square

THEOREM 2.2. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that*

$$(2.9) \quad \Phi(x, y) = \sum_{k=0}^{\infty} 16^k \varphi(2^{-k}x, 2^{-k}y) < \infty$$

for all $x, y \in X$. Suppose that $f : X \rightarrow C_{cb}(Y)$ is the mapping satisfying (2.2) for all $x, y \in X$. Then the mapping $F : X \rightarrow C_{cb}(Y)$ defined by

$$F(x) = \lim_{n \rightarrow \infty} \left(\frac{16^n + 2^n}{2} f(2^{-n}x) \oplus \frac{16^n - 2^n}{2} f(-2^{-n}x) \right)$$

for all $x \in X$, is a unique solution of the functional equation (1.2) such that

$$(2.10) \quad h(f(x), F(x)) \leq \sum_{k=0}^{\infty} \left(\frac{16^k + 2^k}{2} \varphi\left(\frac{x}{2^{k+1}}, 0\right) + \frac{16^k - 2^k}{2} \varphi\left(\frac{-x}{2^{k+1}}, 0\right) \right)$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2^{n+1}}$ in (2.5), I get

$$h\left(f\left(\frac{x}{2^n}\right), 9f\left(\frac{x}{2^{n+1}}\right) \oplus 7f\left(\frac{-x}{2^{n+1}}\right)\right) \leq \varphi\left(\frac{x}{2^{n+1}}, 0\right)$$

for all $x \in X$ and $n \in \mathbb{N}$. By the above inequality and the inequality (i) in Lemma 1.1, I have

$$\begin{aligned}
& h\left(\frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right), \right. \\
& \quad \left. \frac{16^{n+1} + 2^{n+1}}{2}f\left(\frac{x}{2^{n+1}}\right) \oplus \frac{16^{n+1} - 2^{n+1}}{2}f\left(\frac{-x}{2^{n+1}}\right)\right) \\
& \leq h\left(\frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right), \frac{9(16^n + 2^n)}{2}f\left(\frac{x}{2^{n+1}}\right) \oplus \frac{7(16^n + 2^n)}{2}f\left(\frac{-x}{2^{n+1}}\right)\right) \\
& \quad + h\left(\frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right), \frac{9(16^n - 2^n)}{2}f\left(\frac{-x}{2^{n+1}}\right) \oplus \frac{7(16^n - 2^n)}{2}f\left(\frac{x}{2^{n+1}}\right)\right) \\
(2.11) \quad & \leq \frac{16^n + 2^n}{\varphi}\left(\frac{x}{2^{n+1}}, 0\right) + \frac{16^n - 2^n}{2}\varphi\left(\frac{-x}{2^{n+1}}, 0\right)
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. From the above inequality and the property $h(A, C) \leq h(A, B) + h(B, C)$,

$$\begin{aligned}
& h\left(f(x), \frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right)\right) \\
& \leq \sum_{k=0}^{n-1} h\left(\frac{16^k + 2^k}{2}f\left(\frac{x}{2^k}\right) \oplus \frac{16^k - 2^k}{2}f\left(\frac{-x}{2^k}\right), \right. \\
& \quad \left. \frac{16^{k+1} + 8^{k+1}}{2}f\left(\frac{x}{2^{k+1}}\right) \oplus \frac{16^{k+1} - 8^{k+1}}{2}f\left(\frac{-x}{2^{k+1}}\right)\right) \\
(2.12) \quad & \leq \sum_{k=0}^{n-1} \left(\frac{16^k + 2^k}{2}\varphi\left(\frac{x}{2^{k+1}}, 0\right) + \frac{16^k - 2^k}{2}\varphi\left(\frac{-x}{2^{k+1}}, 0\right)\right)
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. In particular, if f is a solution of the functional equation od (1.2), then I know that

$$(2.13) \quad f(x) = \frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Now I claim that the sequence $\left\{\frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right)\right\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$. Indeed, for all

$m, n \in \mathbb{N}$, by (2.12), I know

$$\begin{aligned} & h\left(\frac{16^m + 2^m}{2}f\left(\frac{x}{2^m}\right) \oplus \frac{16^m - 2^m}{2}f\left(\frac{-x}{2^m}\right), \right. \\ & \quad \left. \frac{16^{m+n} + 2^{m+n}}{2}f\left(\frac{x}{2^{m+n}}\right) \oplus \frac{16^{m+n} - 2^{m+n}}{2}f\left(\frac{-x}{2^{m+n}}\right)\right) \\ & \leq \sum_{k=m}^{m+n-1} \left(\frac{16^k + 2^k}{2}\varphi\left(\frac{x}{2^{k+1}}, 0\right) + \frac{16^k - 2^k}{2}\varphi\left(\frac{-x}{2^{k+1}}, 0\right) \right) \end{aligned}$$

for all $x \in X$. It follows from the condition (2.9) that the right hand side of the above inequality tends to zero as $m \rightarrow \infty$. So the sequence $\left\{ \frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right) \right\}$ is a Cauchy sequence. Since $C_{cb}(Y)$ is complete, I can define a set valued mapping $F : X \rightarrow C_{cb}(Y)$ represented by

$$F(x) = \lim_{n \rightarrow \infty} \left(\frac{16^n + 2^n}{2}f(2^{-n}x) \oplus \frac{16^n - 2^n}{2}f(-2^{-n}x) \right)$$

for all $x \in X$. Next, I will show that F satisfies the the functional equation (1.2). Similar to Theorem 2.1, the inequality

$$\begin{aligned} & h\left(\frac{16^n + 2^n}{2}f\left(\frac{x+3y}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x-3y}{2^n}\right) \right. \\ & \quad \oplus \frac{10(16^n + 2^n)}{2}f\left(\frac{x+y}{2^n}\right) \oplus \frac{10(16^n - 2^n)}{2}f\left(\frac{-x-y}{2^n}\right) \\ & \quad \oplus \frac{7(16^n + 2^n)}{2}f\left(\frac{-x}{2^n}\right) \oplus \frac{7(16^n - 2^n)}{2}f\left(\frac{x}{2^n}\right) \\ & \quad \oplus \frac{5(16^n + 2^n)}{2}f\left(\frac{x-y}{2^n}\right) \oplus \frac{5(16^n - 2^n)}{2}f\left(\frac{-x+y}{2^n}\right), \\ & \quad \frac{5(16^n + 2^n)}{2}f\left(\frac{x+2y}{2^n}\right) \oplus \frac{5(16^n - 2^n)}{2}f\left(\frac{-x-2y}{2^n}\right) \\ & \quad \oplus \frac{16^n + 2^n}{2}f\left(\frac{x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x}{2^n}\right) \\ & \quad \oplus \frac{16^n + 2^n}{2}f\left(\frac{2x}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-2x}{2^n}\right) \\ & \quad \left. \oplus \frac{16^n + 2^n}{2}f\left(\frac{x-2y}{2^n}\right) \oplus \frac{16^n - 2^n}{2}f\left(\frac{-x+2y}{2^n}\right) \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{16^n + 2^n}{2} h \left(f \left(\frac{x+3y}{2^n} \right) \oplus 10f \left(\frac{x+y}{2^n} \right) \oplus 7f \left(\frac{-x}{2^n} \right) \oplus 5f \left(\frac{x-y}{2^n} \right), \right. \\
&\quad \left. 5f \left(\frac{x+2y}{2^n} \right) \oplus f \left(\frac{x}{2^n} \right) \oplus f \left(\frac{2x}{2^n} \right) \oplus f \left(\frac{x-2y}{2^n} \right) \right) \\
&\quad + \frac{16^n - 2^n}{2} h \left(f \left(\frac{-x-3y}{2^n} \right) \oplus 10f \left(\frac{-x-y}{2^n} \right) \oplus 7f \left(\frac{x}{2^n} \right) \right. \\
&\quad \left. \oplus 5f \left(\frac{-x+y}{2^n} \right), f \left(\frac{-x-2y}{2^n} \right) \oplus f \left(\frac{-x}{2^n} \right) \right. \\
&\quad \left. \oplus f \left(\frac{-2x}{2^n} \right) \oplus f \left(\frac{-x+2y}{2^n} \right) \right) \\
&\leq \frac{16^n + 2^n}{2} \varphi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) + \frac{16^n - 2^n}{2} \varphi \left(\frac{-x}{2^n}, \frac{-y}{2^n} \right)
\end{aligned}$$

for all $x, y \in X$, leads us that F satisfies the the functional equation (1.2). Moreover, letting $n \rightarrow \infty$ in (2.12), I get the desired inequality (2.10). To prove the uniqueness of F , assume that a set-valued mapping F' is another solution of (1.2) satisfying the inequality (2.10). Then the equality $F'(x) = \frac{16^n+2^n}{2} F' \left(\frac{x}{2^n} \right) \oplus \frac{16^n-2^n}{2} F' \left(\frac{-x}{2^n} \right)$ follows from the equality (2.13). Thus I have

$$\begin{aligned}
&h \left(F'(x), \frac{16^n + 2^n}{2} f \left(\frac{x}{2^n} \right) \oplus \frac{16^n - 2^n}{2} f \left(\frac{-x}{2^n} \right) \right) \\
&= h \left(\frac{16^n + 2^n}{2} F' \left(\frac{x}{2^n} \right) \oplus \frac{16^n - 2^n}{2} F' \left(\frac{-x}{2^n} \right), \right. \\
&\quad \left. \frac{16^n + 2^n}{2} f \left(\frac{x}{2^n} \right) \oplus \frac{16^n - 2^n}{2} f \left(\frac{-x}{2^n} \right) \right) \\
&\leq \frac{16^n + 2^n}{2} h \left(F' \left(\frac{x}{2^n} \right), f \left(\frac{x}{2^n} \right) \right) + \frac{16^n - 2^n}{2} h \left(F' \left(\frac{-x}{2^n} \right), f \left(\frac{-x}{2^n} \right) \right) \\
&\leq (16^n + 2^n) \sum_{k=0}^{\infty} \left(\frac{16^k + 2^k}{2} \varphi \left(\frac{x}{2^{k+n+1}}, 0 \right) + \frac{16^k - 2^k}{2} \varphi \left(\frac{-x}{2^{k+n+1}}, 0 \right) \right) \\
&\quad + (16^n - 2^n) \sum_{k=0}^{\infty} \left(\frac{16^k + 2^k}{2} \varphi \left(\frac{-x}{2^{k+n+1}}, 0 \right) + \frac{16^k - 2^k}{2} \varphi \left(\frac{x}{2^{k+n+1}}, 0 \right) \right) \\
&\leq 2 \cdot 16^n \sum_{k=0}^{\infty} 16^k \left(\varphi \left(\frac{x}{2^{k+n+1}}, 0 \right) + \varphi \left(\frac{-x}{2^{k+n+1}}, 0 \right) \right)
\end{aligned}$$

$$\leq \sum_{k=n+1}^{\infty} 16^k \left(\varphi \left(\frac{x}{2^k}, 0 \right) + \varphi \left(\frac{-x}{2^k}, 0 \right) \right) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which means that $F'(x) = \lim_{n \rightarrow \infty} \frac{16^n + 2^n}{2} f \left(\frac{x}{2^n} \right) \oplus \frac{16^n - 2^n}{2} f \left(\frac{-x}{2^n} \right)$ for all $x \in X$. Therefore $F(x) = F'(x)$ holds for all $x \in X$. This completes the proof of the theorem. \square

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